SUPPLEMENT 3A

FACTORING POLYNOMIALS

Scientists use factoring to calculate growth rates of infectious diseases such as viruses. (credit: “FotoshopTofs” / Pixabay)

Chapter Outline

Introduction

An epidemic of a disease has broken out. Where did it start? How is it spreading? What can be done to control it? Answers to these and other questions can be found by scientists known as epidemiologists. They collect data and analyze it to study disease and consider possible control measures. Because diseases can spread at alarming rates, these scientists must use their knowledge of mathematics involving factoring. In this chapter, you will learn how to factor and apply factoring to real-life situations.

6.1

Greatest Common Factor and Factor by Grouping

Learning Objectives

By the end of this section, you will be able to:

- Find the greatest common factor of two or more expressions
- Factor the greatest common factor from a polynomial
- Factor by grouping

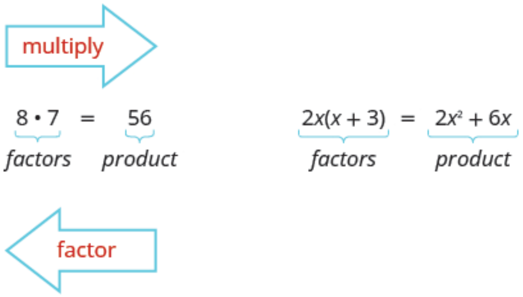
Be Prepared!

Before you get started, take this readiness quiz.

1. Factor 56 into primes.
2. Find the least common multiple (LCM) of 18 and 24.
3. Multiply: $-3a(7a + 8b)$.

Find the Greatest Common Factor of Two or More Expressions

Earlier we multiplied factors together to get a product. Now, we will reverse this process; we will start with a product and then break it down into its factors. Splitting a product into factors is called **factoring**.



We have learned how to factor numbers to find the least common multiple (LCM) of two or more numbers. Now we will factor expressions and find the **greatest common factor** of two or more expressions. The method we use is similar to what we used to find the LCM.

Greatest Common Factor

The **greatest common factor** (GCF) of two or more expressions is the largest expression that is a factor of all the expressions.

We summarize the steps we use to find the greatest common factor.

HOW TO : : FIND THE GREATEST COMMON FACTOR (GCF) OF TWO EXPRESSIONS.

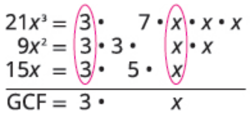
Step 1.	Factor each coefficient into primes. Write all variables with exponents in expanded form.
Step 2.	List all factors—matching common factors in a column. In each column, circle the common factors.
Step 3.	Bring down the common factors that all expressions share.
Step 4.	Multiply the factors.

The next example will show us the steps to find the greatest common factor of three expressions.

EXAMPLE 6.1

Find the greatest common factor of $21x^3,\ 9x^2,\ 15x$.

Solution

Factor each coefficient into primes and write the variables with exponents in expanded form. Circle the common factors in each column. Bring down the common factors.	
Multiply the factors.	GCF = $3x$
	The GCF of $21x^3$, $9x^2$ and $15x$ is $3x$.

TRY IT : : 6.1 Find the greatest common factor: $25m^4,\ 35m^3,\ 20m^2$.

TRY IT : : 6.2 Find the greatest common factor: $14x^3,\ 70x^2,\ 105x$.

Factor the Greatest Common Factor from a Polynomial

It is sometimes useful to represent a number as a product of factors, for example, 12 as $2 \cdot 6$ or $3 \cdot 4$. In algebra, it can also be useful to represent a polynomial in factored form. We will start with a product, such as $3x^2 + 15x$, and end with its factors, $3x(x+5)$. To do this we apply the Distributive Property “in reverse.”

We state the Distributive Property here just as you saw it in earlier chapters and “in reverse.”

Distributive Property

If *a*, *b*, and *c* are real numbers, then

$$a(b+c) = ab + ac \quad \text{and} \quad ab + ac = a(b+c)$$

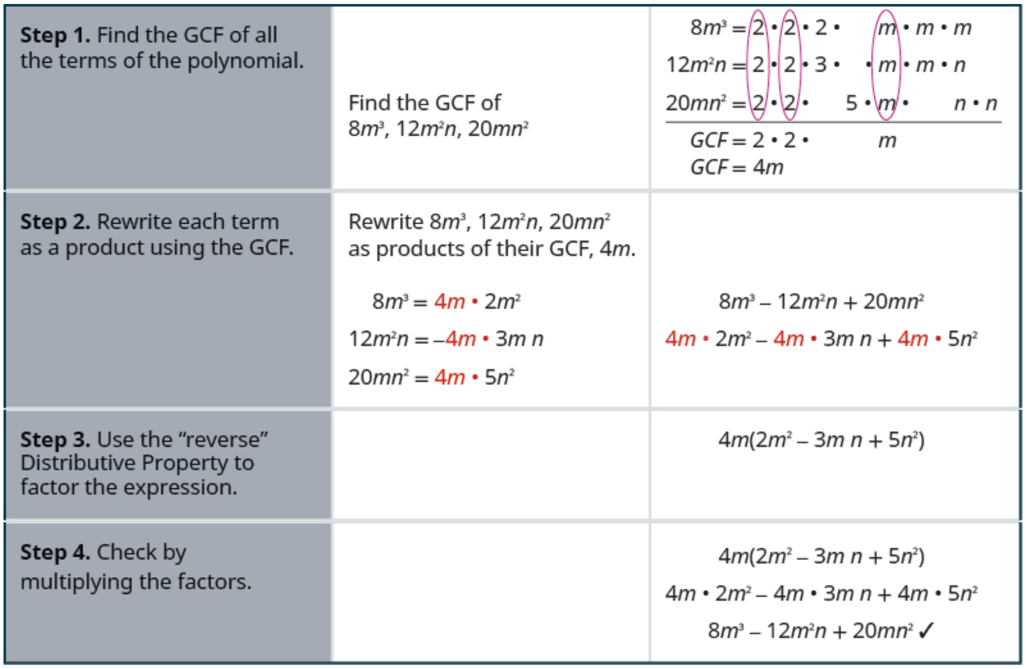
The form on the left is used to multiply. The form on the right is used to factor.

So how do you use the Distributive Property to factor a polynomial? You just find the GCF of all the terms and write the polynomial as a product!

EXAMPLE 6.2 HOW TO USE THE DISTRIBUTIVE PROPERTY TO FACTOR A POLYNOMIAL

Factor: $8m^3 - 12m^2n + 20mn^2$.

Solution



Step 1. Find the GCF of all the terms of the polynomial.	Find the GCF of $8m^3$, $12m^2n$, $20mn^2$	$8m^3 = 2 \cdot 2 \cdot 2 \cdot m \cdot m \cdot m$ $12m^2n = 2 \cdot 2 \cdot 3 \cdot m \cdot m \cdot n$ $20mn^2 = 2 \cdot 2 \cdot 5 \cdot m \cdot n \cdot n$ $GCF = 2 \cdot 2 \cdot m$ $GCF = 4m$
Step 2. Rewrite each term as a product using the GCF.	Rewrite $8m^3$, $12m^2n$, $20mn^2$ as products of their GCF, $4m$. $8m^3 = 4m \cdot 2m^2$ $12m^2n = -4m \cdot 3m\,n$ $20mn^2 = 4m \cdot 5n^2$	$8m^3 - 12m^2n + 20mn^2$ $4m \cdot 2m^2 - 4m \cdot 3m\,n + 4m \cdot 5n^2$
Step 3. Use the “reverse” Distributive Property to factor the expression.		$4m(2m^2 - 3m\,n + 5n^2)$
Step 4. Check by multiplying the factors.		$4m(2m^2 - 3m\,n + 5n^2)$ $4m \cdot 2m^2 - 4m \cdot 3m\,n + 4m \cdot 5n^2$ $8m^3 - 12m^2n + 20mn^2$ ✓

TRY IT : : 6.3 Factor: $9xy^2 + 6x^2y^2 + 21y^3$.

TRY IT : : 6.4 Factor: $3p^3 - 6p^2q + 9pq^3$.

HOW TO : : FACTOR THE GREATEST COMMON FACTOR FROM A POLYNOMIAL.

Step 1. Find the GCF of all the terms of the polynomial.
Step 2. Rewrite each term as a product using the GCF.
Step 3. Use the “reverse” Distributive Property to factor the expression.
Step 4. Check by multiplying the factors.

Factor as a Noun and a Verb

We use “factor” as both a noun and a verb:

Noun: 7 is a *factor* of 14
Verb: *factor* 3 from $3a + 3$

EXAMPLE 6.3

Factor: $5x^3 - 25x^2$.

Solution

Find the GCF of $5x^3$ and $25x^2$.	$5x^3 = \enclose{circle}{5} \cdot \enclose{circle}{x} \cdot \enclose{circle}{x} \cdot x$ $25x^2 = \enclose{circle}{5} \cdot 5 \cdot \enclose{circle}{x} \cdot \enclose{circle}{x}$ $\text{GCF} = 5 \cdot x \cdot x$
	$\text{GCF} = 5x^2$
	$5x^3 - 25x^2$
Rewrite each term.	$5x^2 \cdot x - 5x^2 \cdot 5$
Factor the GCF.	$5x^2(x - 5)$

Check:

$$5x^2(x-5)$$
$$5x^2 \cdot x - 5x^2 \cdot 5$$
$$5x^3 - 25x^2 \checkmark$$

TRY IT : : 6.5 Factor: $2x^3 + 12x^2$.

TRY IT : : 6.6 Factor: $6y^3 - 15y^2$.

EXAMPLE 6.4

Factor: $8x^3y - 10x^2y^2 + 12xy^3$.

Solution

The GCF of $8x^3y,\ -10x^2y^2,$ and $12xy^3$ is $2xy$.	$\begin{array}{rclll} 8x^3y & = & 2\cdot2\cdot2 & x\cdot x\cdot x\cdot & y \\ 10x^2y^2 & = & 2\cdot\quad 5 & x\cdot x\cdot & y\cdot y \\ 12xy^3 & = & 2\cdot2\cdot3 & x\cdot & y\cdot y\cdot y \\ \hline \text{GCF} & = & 2\cdot & x\cdot & y \end{array}$
	$\text{GCF} = 2xy$
	$8x^3y - 10x^2y^2 + 12xy^3$
Rewrite each term using the GCF, $2xy$.	$2xy \cdot 4x^2 - 2xy \cdot 5xy + 2xy \cdot 6y^2$
Factor the GCF.	$2xy(4x^2 - 5xy + 6y^2)$

Check:

$$2xy\left(4x^2 - 5xy + 6y^2\right)$$
$$2xy \cdot 4x^2 - 2xy \cdot 5xy + 2xy \cdot 6y^2$$
$$8x^3y - 10x^2y^2 + 12xy^3 \checkmark$$

TRY IT : : 6.7 Factor: $15x^3y - 3x^2y^2 + 6xy^3$.

TRY IT : : 6.8 Factor: $8a^3b + 2a^2b^2 - 6ab^3$.

When the leading coefficient is negative, we factor the negative out as part of the GCF.

EXAMPLE 6.5

Factor: $-4a^3 + 36a^2 - 8a$.

Solution

The leading coefficient is negative, so the GCF will be negative.

	$-4a^3 + 36a^2 - 8a$
Rewrite each term using the GCF, $-4a$.	$-4a \cdot a^2 - (-4a) \cdot 9a + (-4a) \cdot 2$
Factor the GCF.	$-4a(a^2 - 9a + 2)$

Check:

$$-4a\left(a^2 - 9a + 2\right)$$
$$-4a \cdot a^2 - (-4a) \cdot 9a + (-4a) \cdot 2$$
$$-4a^3 + 36a^2 - 8a \checkmark$$

TRY IT : : 6.9 Factor: $-4b^3 + 16b^2 - 8b$.

TRY IT : : 6.10 Factor: $-7a^3 + 21a^2 - 14a$.

So far our greatest common factors have been monomials. In the next example, the greatest common factor is a binomial.

EXAMPLE 6.6

Factor: $3y(y+7)-4(y+7)$.

Solution

The GCF is the binomial $y+7$.

	$3y(y+7)-4(y+7)$
Factor the GCF, $(y+7)$.	$(y+7)(3y-4)$
Check on your own by multiplying.	

TRY IT : : 6.11 Factor: $4m(m+3)-7(m+3)$.

TRY IT : : 6.12 Factor: $8n(n-4)+5(n-4)$.

Factor by Grouping

Sometimes there is no common factor of all the terms of a polynomial. When there are four terms we separate the polynomial into two parts with two terms in each part. Then look for the GCF in each part. If the polynomial can be factored, you will find a common factor emerges from both parts. Not all polynomials can be factored. Just like some numbers are prime, some polynomials are prime.

EXAMPLE 6.7 HOW TO FACTOR A POLYNOMIAL BY GROUPING

Factor by grouping: $xy+3y+2x+6$.

Solution

Step 1. Group terms with common factors.	Is there a greatest common factor of all four terms?	$xy+3y+2x+6$
	No, so let's separate the first two terms from the second two.	$\underbrace{xy+3y}+\underbrace{2x+6}$
Step 2. Factor out the common factor in each group.	Factor the GCF from the first two terms.	$y(x+3)+\underbrace{2x+6}$
	Factor the GCF from the second two terms.	$y(x+3)+2(x+3)$
Step 3. Factor the common factor from the expression.	Notice that each term has a common factor of $(x+3)$.	$y(x+3)+2(x+3)$
	Factor out the common factor.	$(x+3)(y+2)$
Step 4. Check.	Multiply $(x+3)(y+2)$. Is the product the original expression?	$(x+3)(y+2)$ $xy+2x+3y+6$ $xy+3y+2x+6$ ✓

TRY IT : : 6.13 Factor by grouping: $xy+8y+3x+24$.

TRY IT : : 6.14 Factor by grouping: $ab+7b+8a+56$.

HOW TO : : FACTOR BY GROUPING.

Step 1. Group terms with common factors.
Step 2. Factor out the common factor in each group.
Step 3. Factor the common factor from the expression.
Step 4. Check by multiplying the factors.

EXAMPLE 6.8

Factor by grouping: ⓐ $x^2 + 3x - 2x - 6$ ⓑ $6x^2 - 3x - 4x + 2$.

✓ Solution

ⓐ

There is no GCF in all four terms.	$x^2 + 3x - 2x - 6$
Separate into two parts.	$x^2 + 3x \quad -2x - 6$
Factor the GCF from both parts. Be careful with the signs when factoring the GCF from the last two terms.	$x(x + 3) - 2(x + 3)$
Factor out the common factor.	$(x + 3)(x - 2)$
Check on your own by multiplying.	

ⓑ

There is no GCF in all four terms.	$6x^2 - 3x - 4x + 2$
Separate into two parts.	$6x^2 - 3x \quad -4x + 2$
Factor the GCF from both parts.	$3x(2x - 1) - 2(2x - 1)$
Factor out the common factor.	$(2x - 1)(3x - 2)$
Check on your own by multiplying.	

TRY IT : : 6.15 Factor by grouping: ⓐ $x^2 + 2x - 5x - 10$ ⓑ $20x^2 - 16x - 15x + 12$.

TRY IT : : 6.16 Factor by grouping: ⓐ $y^2 + 4y - 7y - 28$ ⓑ $42m^2 - 18m - 35m + 15$.

6.1 EXERCISES

Practice Makes Perfect

Find the Greatest Common Factor of Two or More Expressions

In the following exercises, find the greatest common factor.

1. $10p^3q,\ 12pq^2$

2. $8a^2b^3,\ 10ab^2$

3. $12m^2n^3,\ 30m^5n^3$

4. $28x^2y^4,\ 42x^4y^4$

5. $10a^3,\ 12a^2,\ 14a$

6. $20y^3,\ 28y^2,\ 40y$

7. $35x^3y^2,\ 10x^4y,\ 5x^5y^3$

8. $27p^2q^3,\ 45p^3q^4,\ 9p^4q^3$

Factor the Greatest Common Factor from a Polynomial

In the following exercises, factor the greatest common factor from each polynomial.

9. $6m+9$

11. $9n-63$

12. $45b-18$

14. $4y^2+8y-4$

15. $8p^2+4p+2$

17. $8y^3+16y^2$

18. $12x^3-10x$

20. $8m^2-40m+16$

21. $24x^3-12x^2+15x$

23. $12xy^2+18x^2y^2-30y^3$

24. $21pq^2+35p^2q^2-28q^3$

26. $24a^3b+6a^2b^2-18ab^3$

27. $-2x-4$

29. $-2x^3+18x^2-8x$

30. $-5y^3+35y^2-15y$

32. $-6a^3b-12a^2b^2+18ab^2$

33. $5x(x+1)+3(x+1)$

35. $3b(b-2)-13(b-2)$

Factor by Grouping

In the following exercises, factor by grouping.

37. $ab+5a+3b+15$

39. $8y^2+y+40y+5$

40. $6y^2+7y+24y+28$

42. $pq-10p+8q-80$

43. $u^2-u+6u-6$

45. $9p^2-3p-20$

46. $16q^2-8q-35$

48. $r^2-3r-r+3$

49. $2x^2-14x-5x+35$

6.2 Factor Trinomials

Learning Objectives

By the end of this section, you will be able to:

- Factor trinomials of the form $x^2 + bx + c$
- Factor trinomials of the form $ax^2 + bx + c$ using trial and error
- Factor trinomials of the form $ax^2 + bx + c$ using the ‘ac’ method
- Factor using substitution

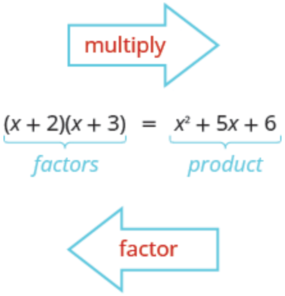
Be Prepared!

Before you get started, take this readiness quiz.

1. Find all the factors of 72.
2. Find the product: $(3y + 4)(2y + 5)$.
3. Simplify: $-9(6); -9(-6)$.

Factor Trinomials of the Form $x^2 + bx + c$

You have already learned how to multiply binomials using FOIL. Now you’ll need to “undo” this multiplication. To factor the trinomial means to start with the product, and end with the factors.



To figure out how we would factor a trinomial of the form $x^2 + bx + c$, such as $x^2 + 5x + 6$ and factor it to $(x + 2)(x + 3)$, let’s start with two general binomials of the form $(x + m)$ and $(x + n)$.

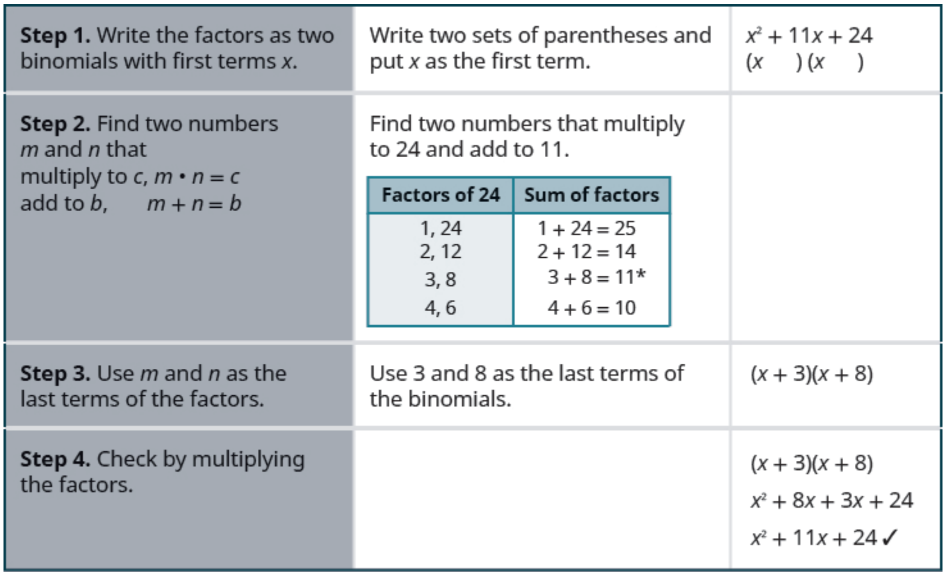
	$(x + m)(x + n)$
Foil to find the product.	$x^2 + mx + nx + mn$
Factor the GCF from the middle terms.	$x^2 + (m + n)x + mn$
Our trinomial is of the form $x^2 + bx + c$.	$\overbrace{x^2 +}^{x^2 +}\overbrace{(m + n)x}^{bx}\overbrace{+ mn}^{+\ c}$

This tells us that to factor a trinomial of the form $x^2 + bx + c$, we need two factors $(x + m)$ and $(x + n)$ where the two numbers *m* and *n* multiply to *c* and add to *b*.

EXAMPLE 6.9 HOW TO FACTOR A TRINOMIAL OF THE FORM $x^2 + bx + c$

Factor: $x^2 + 11x + 24$.

Solution



Step 1. Write the factors as two binomials with first terms *x*.	Write two sets of parentheses and put *x* as the first term.	$x^2 + 11x + 24$ $(x \quad)(x \quad)$
Step 2. Find two numbers *m* and *n* that multiply to *c*, $m \cdot n = c$ add to *b*, $m + n = b$	Find two numbers that multiply to 24 and add to 11.	
Step 3. Use *m* and *n* as the last terms of the factors.	Use 3 and 8 as the last terms of the binomials.	$(x + 3)(x + 8)$
Step 4. Check by multiplying the factors.		$(x + 3)(x + 8)$ $x^2 + 8x + 3x + 24$ $x^2 + 11x + 24$ ✓

Factors of 24	Sum of factors
1, 24	$1 + 24 = 25$
2, 12	$2 + 12 = 14$
3, 8	$3 + 8 = 11$*
4, 6	$4 + 6 = 10$

TRY IT : : 6.17 Factor: $q^2 + 10q + 24$.

TRY IT : : 6.18 Factor: $t^2 + 14t + 24$.

Let's summarize the steps we used to find the factors.

HOW TO : : FACTOR TRINOMIALS OF THE FORM $x^2 + bx + c$.

Step 1. Write the factors as two binomials with first terms *x*. $x^2 + bx + c$ $(x \quad)(x \quad)$

Step 2. Find two numbers *m* and *n* that
- multiply to $c,\ m \cdot n = c$
- add to $b,\ m + n = b$

Step 3. Use *m* and *n* as the last terms of the factors. $(x + m)(x + n)$

Step 4. Check by multiplying the factors.

In the first example, all terms in the trinomial were positive. What happens when there are negative terms? Well, it depends which term is negative. Let's look first at trinomials with only the middle term negative.

How do you get a *positive product* and a *negative sum*? We use two negative numbers.

EXAMPLE 6.10

Factor: $y^2 - 11y + 28$.

Solution

Again, with the positive last term, 28, and the negative middle term, $-11y$, we need two negative factors. Find two numbers that multiply 28 and add to -11.

	$y^2 - 11y + 28$
Write the factors as two binomials with fir t terms y.	$(y\quad)(y\quad)$
Find two numbers that: multiply to 28 and add to -11.	

Factors of 28	Sum of factors
$-1, -28$	$-1 + (-28) = -29$
$-2, -14$	$-2 + (-14) = -16$
$-4, -7$	$-4 + (-7) = -11^*$

Use $-4,\ -7$ as the last terms of the binomials.	$(y - 4)(y - 7)$

Check:

$$(y - 4)(y - 7)$$
$$y^2 - 7y - 4y + 28$$
$$y^2 - 11y + 28\ ✓$$

TRY IT : : 6.19 Factor: $u^2 - 9u + 18$.

TRY IT : : 6.20 Factor: $y^2 - 16y + 63$.

Now, what if the last term in the trinomial is negative? Think about FOIL. The last term is the product of the last terms in the two binomials. A negative product results from multiplying two numbers with opposite signs. You have to be very careful to choose factors to make sure you get the correct sign for the middle term, too.

How do you get a *negative product* and a *positive sum*? We use one positive and one negative number.

When we factor trinomials, we must have the terms written in descending order—in order from highest degree to lowest degree.

EXAMPLE 6.11

Factor: $2x + x^2 - 48$.

Solution

$2x + x^2 - 48$

First we put the terms in decreasing degree order. $x^2 + 2x - 48$

Factors will be two binomials with first terms x. $(x \quad)(x \quad)$

Factors of -48	Sum of factors
$-1, 48$	$-1 + 48 = 47$
$-2, 24$	$-2 + 24 = 22$
$-3, 16$	$-3 + 16 = 13$
$-4, 12$	$-4 + 12 = 8$
$-6, 8$	$-6 + 8 = 2^*$

Use $-6, 8$ as the last terms of the binomials. $(x - 6)(x + 8)$

Check:

$$(x - 6)(x + 8)$$
$$x^2 - 6q + 8q - 48$$
$$x^2 + 2x - 48 ✓$$

TRY IT : : 6.21 Factor: $9m + m^2 + 18$.

TRY IT : : 6.22 Factor: $-7n + 12 + n^2$.

Sometimes you'll need to factor trinomials of the form $x^2 + bxy + cy^2$ with two variables, such as $x^2 + 12xy + 36y^2$. The first term, x^2, is the product of the first terms of the binomial factors, $x \cdot x$. The y^2 in the last term means that the second terms of the binomial factors must each contain *y*. To get the coefficients *b* and *c*, you use the same process summarized in **How To Factor trinomials**.

EXAMPLE 6.12

Factor: $r^2 - 8rs - 9s^2$.

Solution

We need r in the first term of each binomial and s in the second term. The last term of the trinomial is negative, so the factors must have opposite signs.

$r^2 - 8rs - 9s^2$

Note that the first terms are r, last terms contain s. $(r \quad s)(r \quad s)$

Find the numbers that multiply to -9 and add to -8.

Factors of -9	Sum of factors
$1, -9$	$-1 + 9 = 8$
$-1, 9$	$1 + (-9) = -8*$
$3, -3$	$3 + (-3) = 0$

Use $1, -9$ as coefficients of he last terms. $(r + s)(r - 9s)$

Check:

$$(r - 9s)(r + s)$$
$$r^2 + rs - 9rs - 9s^2$$
$$r^2 - 8rs - 9s^2 \checkmark$$

TRY IT : : 6.23 Factor: $a^2 - 11ab + 10b^2$.

TRY IT : : 6.24 Factor: $m^2 - 13mn + 12n^2$.

Some trinomials are prime. The only way to be certain a trinomial is prime is to list all the possibilities and show that none of them work.

EXAMPLE 6.13

Factor: $u^2 - 9uv - 12v^2$.

Solution

We need *u* in the first term of each binomial and *v* in the second term. The last term of the trinomial is negative, so the factors must have opposite signs.

$$u^2 - 9uv - 12v^2$$

Note that the fir t terms are u, last terms contain v. $(u \quad v)(u \quad v)$

Find the numbers that multiply to -12 and add to -9.

Factors of -12	Sum of factors
$1, -12$	$1 + (-12) = -11$
$-1, 12$	$-1 + 12 = 11$
$2, -6$	$2 + (-6) = -4$
$-2, 6$	$-2 + 6 = 4$
$3, -4$	$3 + (-4) = -1$
$-3, 4$	$-3 + 4 = 1$

Note there are no factor pairs that give us -9 as a sum. The trinomial is prime.

TRY IT : : 6.25 Factor: $x^2 - 7xy - 10y^2$.

TRY IT : : 6.26 Factor: $p^2 + 15pq + 20q^2$.

Let’s summarize the method we just developed to factor trinomials of the form $x^2 + bx + c$.

Strategy for Factoring Trinomials of the Form $x^2 + bx + c$

When we factor a trinomial, we look at the signs of its terms first to determine the signs of the binomial factors.

$$x^2 + bx + c$$
$$(x + m)(x + n)$$

When c is positive, m and n have the same sign.

b positive	b negative
$m,\ n$ positive	$m,\ n$ negative
$x^2 + 5x + 6$	$x^2 - 6x + 8$
$(x + 2)(x + 3)$	$(x - 4)(x - 2)$
same signs	same signs

When c is negative, m and n have opposite signs.

$x^2 + x - 12$	$x^2 - 2x - 15$
$(x + 4)(x - 3)$	$(x - 5)(x + 3)$
opposite signs	opposite signs

Notice that, in the case when *m* and *n* have opposite signs, the sign of the one with the larger absolute value matches the sign of *b*.

Factor Trinomials of the form $ax^2 + bx + c$ using Trial and Error

Our next step is to factor trinomials whose leading coefficient is not 1, trinomials of the form $ax^2 + bx + c$.

Remember to always check for a GCF first! Sometimes, after you factor the GCF, the leading coefficient of the trinomial becomes 1 and you can factor it by the methods we've used so far. Let's do an example to see how this works.

EXAMPLE 6.14

Factor completely: $4x^3 + 16x^2 - 20x$.

Solution

Is there a greatest common factor?	$4x^3 + 16x^2 - 20x$
Yes, GCF $= 4x$. Factor it.	$4x\left(x^2 + 4x - 5\right)$
Binomial, trinomial, or more than three terms? It is a trinomial. So "undo FOIL."	$4x(x \quad)(x \quad)$
Use a table like the one shown to find t o numbers that multiply to -5 and add to 4.	$4x(x - 1)(x + 5)$

Factors of -5	Sum of factors
$-1,\ 5$	$-1 + 5 = 4^*$
$1,\ -5$	$1 + (-5) = -4$

Check:

$$4x(x - 1)(x + 5)$$
$$4x(x^2 + 5x - x - 5)$$
$$4x(x^2 + 4x - 5)$$
$$4x^3 + 16x^2 - 20x \checkmark$$

TRY IT : : 6.27 Factor completely: $5x^3+15x^2-20x$.

TRY IT : : 6.28 Factor completely: $6y^3+18y^2-60y$.

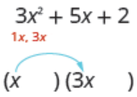
What happens when the leading coefficient is not 1 and there is no GCF? There are several methods that can be used to factor these trinomials. First we will use the Trial and Error method.

Let’s factor the trinomial $3x^2+5x+2$.

From our earlier work, we expect this will factor into two binomials.

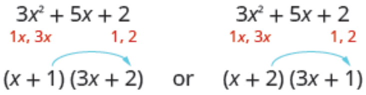
$$\begin{array}{c}3x^2+5x+2\\ (\quad\ )(\quad\ )\end{array}$$

We know the first terms of the binomial factors will multiply to give us $3x^2$. The only factors of $3x^2$ are $1x, 3x$. We can place them in the binomials.

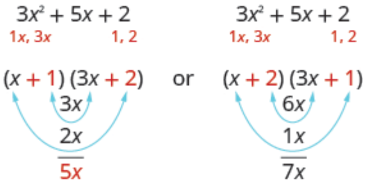


Check: Does $1x\cdot 3x=3x^2$?

We know the last terms of the binomials will multiply to 2. Since this trinomial has all positive terms, we only need to consider positive factors. The only factors of 2 are 1, 2. But we now have two cases to consider as it will make a difference if we write 1, 2 or 2, 1.



Which factors are correct? To decide that, we multiply the inner and outer terms.



Since the middle term of the trinomial is $5x$, the factors in the first case will work. Let’s use FOIL to check.

$$\begin{array}{c}(x+1)(3x+2)\\ 3x^2+2x+3x+2\\ 3x^2+5x+2\ \checkmark\end{array}$$

Our result of the factoring is:

$$\begin{array}{c}3x^2+5x+2\\ (x+1)(3x+2)\end{array}$$

EXAMPLE 6.15 HOW TO FACTOR A TRINOMIAL USING TRIAL AND ERROR

Factor completely using trial and error: $3y^2+22y+7$.

Solution

Step 1. Write the trinomial in descending order.	The trinomial is already in descending order.	$3y^2+22y+7$

<table>
<tr><td>Step 2. Factor any GCF.</td><td>There is no GCF.</td><td></td></tr>
<tr><td>Step 3. Find all the factor pairs of the first term.</td><td>The only of 3y² are 1y, 3y.
Since there is only one pair, we can put them in the parentheses.</td><td>$\underset{1y,\,3y}{3y^2} + 22y + 7$
$\underset{1y,\,3y}{3y^2} + 22y + 7$
$(y \quad)(3y \quad)$</td></tr>
<tr><td>Step 4. Find all the factor pairs of the third term.</td><td>The only factors of 7 are 1, 7.</td><td>$\underset{1y,\,3y}{3y^2} + 22y + \underset{1,\,7}{7}$
$(y \quad)(3y \quad)$</td></tr>
<tr><td>Step 5. Test all the possible combinations of the factors until the correct product is found.</td><td>$\underset{1y,\,3y}{3y^2} + 22y + \underset{1,\,7}{7}$
$(y+1)(3y+7)$
$3y$
$7y$
$10y$ No! We need $22y$
$\underset{1y,\,3y}{3y^2} + 22y + \underset{1,\,7}{7}$
$(y+7)(3y+1)$
$21y$
$+y$
$22y$</td><td><table><tr><th colspan="2">$3y^2 + 22y + 7$</th></tr><tr><th>Possible factors</th><th>Product</th></tr><tr><td>$(y+1)(3y+7)$</td><td>$3y^2 + 10y + 7$</td></tr><tr><td>$(y+7)(3y+1)$</td><td>$3y^2 + 22y + 7$</td></tr></table></td></tr>
<tr><td>Step 6. Check by multiplying.</td><td></td><td>$(y+7)(3y+1)$
$3y^2 + 22y + 7$ ✓</td></tr>
</table>

TRY IT : : 6.29 Factor completely using trial and error: $2a^2 + 5a + 3$.

TRY IT : : 6.30 Factor completely using trial and error: $4b^2 + 5b + 1$.

HOW TO : : FACTOR TRINOMIALS OF THE FORM $ax^2 + bx + c$ USING TRIAL AND ERROR.

Step 1.	Write the trinomial in descending order of degrees as needed.
Step 2.	Factor any GCF.
Step 3.	Find all the factor pairs of the first term.
Step 4.	Find all the factor pairs of the third term.
Step 5.	Test all the possible combinations of the factors until the correct product is found.
Step 6.	Check by multiplying.

Remember, when the middle term is negative and the last term is positive, the signs in the binomials must both be negative.

EXAMPLE 6.16

Factor completely using trial and error: $6b^2 - 13b + 5$.

Solution

The trinomial is already in descending order.	$6b^2 - 13b + 5$
Find the factors of the first term.	$6b^2 - 13b + 5$ $1b \cdot 6b$ $2b \cdot 3b$
Find the factors of the last term. Consider the signs. Since the last term, 5, is positive its factors must both be positive or both be negative. The coefficient of the middle term is negative, so we use the negative factors.	$6b^2 - 13b + 5$ $1b \cdot 6b$ $-1, -5$ $2b \cdot 3b$

Consider all the combinations of factors.

$6b^2 - 13b + 5$	
Possible factors	Product
$(b - 1)(6b - 5)$	$6b^2 - 11b + 5$
$(b - 5)(6b - 1)$	$6b^2 - 31b + 5$
$(2b - 1)(3b - 5)$	$6b^2 - 13b + 5$*
$(2b - 5)(3b - 1)$	$6b^2 - 17b + 5$

The correct factors are those whose product is the original trinomial. $(2b - 1)(3b - 5)$

Check by multiplying:

$$(2b - 1)(3b - 5)$$
$$6b^2 - 10b - 3b + 5$$
$$6b^2 - 13b + 5 \checkmark$$

TRY IT : : 6.31 Factor completely using trial and error: $8x^2 - 13x + 3$.

TRY IT : : 6.32 Factor completely using trial and error: $10y^2 - 37y + 7$.

When we factor an expression, we always look for a greatest common factor first. If the expression does not have a greatest common factor, there cannot be one in its factors either. This may help us eliminate some of the possible factor combinations.

EXAMPLE 6.17

Factor completely using trial and error: $18x^2 - 37xy + 15y^2$.

Solution

The trinomial is already in descending order.	$18x^2 - 37xy + 15y^2$
Find the factors of the first term.	$18x^2 - 37xy + 15y^2$ $1x \cdot 18x$ $2x \cdot 9x$ $3x \cdot 6x$
Find the factors of the last term. Consider the signs. Since 15 is positive and the coefficient of the middle term is negative, we use the negative factors.	$18x^2 - 37xy + 15y^2$ $1x \cdot 18x$ $-1, -5$ $2x \cdot 9x$ $-5, -1$ $3x \cdot 6x$

Consider all the combinations of factors.

$18x^2 - 37xy + 15y^2$	
Possible factors	**Product**
$(x - 1y)(18x - 15y)$	Not an option
$(x - 15y)(18x - 1y)$	$18x^2 - 271xy + 15y^2$
$(x - 3y)(18x - 5y)$	$18x^2 - 59xy + 15y^2$
$(x - 5y)(18x - 3y)$	Not an option
$(2x - 1y)(9x - 15y)$	Not an option
$(2x - 15y)(9x - 1y)$	$18x^2 - 137xy + 15y^2$
$(2x - 3y)(9x - 5y)$	$18x^2 - 37xy + 15y^2$*
$(2x - 5y)(9x - 3y)$	Not an option
$(3x - 1y)(6x - 15y)$	Not an option
$(3x - 15y)(6x - 1y)$	Not an option
$(3x - 3y)(6x - 5y)$	Not an option

If the trinomial has no common factors, then neither factor can contain a common factor. That means this combination is not an option.

The correct factors are those whose product is the original trinomial. $(2x - 3y)(9x - 5y)$

Check by multiplying:

$$\begin{gathered}(2x - 3y)(9x - 5y) \\ 18x^2 - 10xy - 27xy + 15y^2 \\ 18x^2 - 37xy + 15y^2 \checkmark\end{gathered}$$

TRY IT : : 6.33 Factor completely using trial and error $18x^2 - 3xy - 10y^2$.

TRY IT : : 6.34 Factor completely using trial and error: $30x^2 - 53xy - 21y^2$.

Don’t forget to look for a GCF first and remember if the leading coefficient is negative, so is the GCF.

EXAMPLE 6.18

Factor completely using trial and error: $-10y^4 - 55y^3 - 60y^2$.

Solution

	$-10y^4 - 55y^3 - 60y^2$
Notice the greatest common factor, so factor it first.	$-5y^2(2y^2 + 11y + 12)$
Factor the trinomial.	$-5y^2\left(\underset{y \cdot 2y}{2y^2} + 11y + \underset{\substack{1 \cdot 12 \\ 2 \cdot 6 \\ 3 \cdot 4}}{12}\right)$

Consider all the combinations.

$2y^2 + 11y + 12$	
Possible factors	**Product**
$(y + 1)(2y + 12)$	Not an option
$(y + 12)(2y + 1)$	$2y^2 + 25y + 12$
$(y + 2)(2y + 6)$	Not an option
$(y + 6)(2y + 2)$	Not an option
$(y + 3)(2y + 4)$	Not an option
$(y + 4)(2y + 3)$	$2y^2 + 11y + 12$*

If the trinomial has no common factors, then neither factor can contain a common factor. That means this combination is not an option.

The correct factors are those whose product is the original trinomial. Remember to include the factor $-5y^2$.

$$-5y^2(y + 4)(2y + 3)$$

Check by multiplying:

$$-5y^2(y + 4)(2y + 3)$$

$$-5y^2\left(2y^2 + 8y + 3y + 12\right)$$

$$-10y^4 - 55y^3 - 60y^2 \checkmark$$

TRY IT : : 6.35 Factor completely using trial and error: $15n^3 - 85n^2 + 100n$.

TRY IT : : 6.36 Factor completely using trial and error: $56q^3 + 320q^2 - 96q$.

Factor Trinomials of the Form $ax^2 + bx + c$ using the "ac" Method

Another way to factor trinomials of the form $ax^2 + bx + c$ is the "ac" method. (The "ac" method is sometimes called the grouping method.) The "ac" method is actually an extension of the methods you used in the last section to factor trinomials with leading coefficient one. This method is very structured (that is step-by-step), and it always works!

EXAMPLE 6.19 HOW TO FACTOR TRINOMIALS USING THE "AC" METHOD

Factor using the *'ac'* method: $6x^2 + 7x + 2$.

Solution

Step 1. Factor any GCF.	Is there a greatest common factor? No!	$6x^2 + 7x + 2$
Step 2. Find the product *ac*.	$a \cdot c$ $6 \cdot 2$ 12	ax^2 + bx + c $6x^2 + 7x + 2$

Step 3. Find two numbers m and n that: Multiply to ac. $m \cdot n = a \cdot c$ Add to b. $m + n = b$	Find two numbers that multiply to 12 and add to 7. Both factors must be positive. $3 \cdot 4 = 12 \quad 3 + 4 = 7$	
Step 4. Split the middle term using m and n. $ax^2 + bx + c$ $ax^2 + \overbrace{mx + nx}^{bx} + c$	Rewrite $7x$ as $3x + 4x$. It would also give the same result if we used $4x + 3x$. Notice that $6x^2 + 3x + 4x + 2$ is equal to $6x^2 + 7x + 2$. We just split the middle term to get a more useful form.	$6x^2 + 7x + 2$ $\underbrace{6x^2 + 3x} + \underbrace{4x + 2}$
Step 5. Factor by grouping.		$3x(2x + 1) + 2(2x + 1)$ $(2x + 1)(3x + 2)$
Step 6. Check by multiplying the factors.		$(2x + 1)(3x + 2)$ $6x^2 + 4x + 3x + 2$ $6x^2 + 7x + 2$ ✓

TRY IT : : 6.37 Factor using the ‘ac’ method: $6x^2 + 13x + 2$.

TRY IT : : 6.38 Factor using the ‘ac’ method: $4y^2 + 8y + 3$.

The “ac” method is summarized here.

HOW TO : : FACTOR TRINOMIALS OF THE FORM $ax^2 + bx + c$ USING THE “AC” METHOD.

Step 1. Factor any GCF.

Step 2. Find the product ac.

Step 3. Find two numbers m and n that:

Multiply to ac $\quad m \cdot n = a \cdot c$

Add to b $\quad m + n = b$

$ax^2 + bx + c$

Step 4. Split the middle term using m and n. $\quad ax^2 + mx + nx + c$

Step 5. Factor by grouping.

Step 6. Check by multiplying the factors.

Don’t forget to look for a common factor!

EXAMPLE 6.20

Factor using the *‘ac’* method: $10y^2 - 55y + 70$.

Solution

Is there a greatest common factor?		
Yes. The GCF is 5.		$10y^2 - 55y + 70$
Factor it.		$5(2y^2 - 11y + 14)$
The trinomial inside the parentheses has a leading coefficient that is not 1.		$5(\overset{ax^2}{2y^2} \overset{+\ bx}{- 11y} \overset{+\ c}{+ 14})$
Find the product ac.	$ac = 28$	
Find two numbers that multiply to ac	$(-4)(-7) = 28$	
and add to *b*.	$-4 + (-7) = -11$	
Split the middle term.		$5(2y^2 - \underset{\swarrow\ \searrow}{11y} + 14)$
		$5(\underbrace{2y^2 - 7y}\ \underbrace{- 4y + 14})$
Factor the trinomial by grouping.		$5(y\,(2y - 7) - 2(y - 7))$
		$5(y - 2)(2y - 7)$

Check by multiplying all three factors.

$$5(y - 2)(2y - 7)$$
$$5\left(2y^2 - 7y - 4y + 14\right)$$
$$5\left(2y^2 - 11y + 14\right)$$
$$10y^2 - 55y + 70\ ✓$$

TRY IT : : 6.39 Factor using the 'ac' method: $16x^2 - 32x + 12.$

TRY IT : : 6.40 Factor using the 'ac' method: $18w^2 - 39w + 18.$

Factor Using Substitution

Sometimes a trinomial does not appear to be in the $ax^2 + bx + c$ form. However, we can often make a thoughtful substitution that will allow us to make it fit the $ax^2 + bx + c$ form. This is called factoring by substitution. It is standard to use *u* for the substitution.

In the $ax^2 + bx + c,$ the middle term has a variable, *x*, and its square, $x^2,$ is the variable part of the first term. Look for this relationship as you try to find a substitution.

EXAMPLE 6.21

Factor by substitution: $x^4 - 4x^2 - 5.$

Solution

The variable part of the middle term is x^2 and its square, $x^4,$ is the variable part of the first term. (We know $\left(x^2\right)^2 = x^4$). If we let $u = x^2,$ we can put our trinomial in the $ax^2 + bx + c$ form we need to factor it.

	x^4-4x^2-5
Rewrite the trinomial to prepare for the substitution.	$(x^2)^2-4(x^2)-5$
Let $u=x^2$ and substitute.	u^2-4u-5
Factor the trinomial.	$(u+1)(u-5)$
Replace *u* with x^2.	$(x^2+1)(x^2-5)$
Check: $\left(x^2+1\right)\left(x^2-5\right)$ $x^4-5x^2+x^2-5$ x^4-4x^2-5 ✓	

TRY IT : : 6.41 Factor by substitution: h^4+4h^2-12.

TRY IT : : 6.42 Factor by substitution: y^4-y^2-20.

Sometimes the expression to be substituted is not a monomial.

EXAMPLE 6.22

Factor by substitution: $(x-2)^2+7(x-2)+12$

Solution

The binomial in the middle term, $(x-2)$ is squared in the first term. If we let $u=x-2$ and substitute, our trinomial will be in ax^2+bx+c form.

	$(x-2)^2+7(x-2)+12$
Rewrite the trinomial to prepare for the substitution.	$(x-2)^2+7(x-2)+12$
Let $u=x-2$ and substitute.	$u^2+7u+12$
Factor the trinomial.	$(u+3)(u+4)$
Replace *u* with $x-2$.	$((x-2)+3)((x-2)+4)$
Simplify inside the parentheses.	$(x+1)(x+2)$

This could also be factored by first multiplying out the $(x-2)^2$ and the $7(x-2)$ and then combining like terms and then factoring. Most students prefer the substitution method.

TRY IT : : 6.43 Factor by substitution: $(x-5)^2+6(x-5)+8$.

TRY IT : : 6.44 Factor by substitution: $(y-4)^2+8(y-4)+15$.

6.2 EXERCISES

Practice Makes Perfect

Factor Trinomials of the Form $x^2 + bx + c$

In the following exercises, factor each trinomial of the form $x^2 + bx + c$.

61. $p^2 + 11p + 30$

63. $n^2 + 19n + 48$

64. $b^2 + 14b + 48$

66. $u^2 + 101u + 100$

67. $x^2 - 8x + 12$

69. $y^2 - 18x + 45$

70. $m^2 - 13m + 30$

72. $y^2 - 5y + 6$

73. $5p - 6 + p^2$

75. $8 - 6x + x^2$

76. $7x + x^2 + 6$

78. $-11 - 10x + x^2$

In the following exercises, factor each trinomial of the form $x^2 + bxy + cy^2$.

79. $x^2 - 2xy - 80y^2$

81. $m^2 - 64mn - 65n^2$

82. $p^2 - 2pq - 35q^2$

84. $r^2 + 3rs - 28s^2$

85. $x^2 - 3xy - 14y^2$

87. $m^2 - 5mn + 30n^2$

88. $c^2 - 7cd + 18d^2$

Factor Trinomials of the Form $ax^2 + bx + c$ Using Trial and Error

In the following exercises, factor completely using trial and error.

89. $p^3 - 8p^2 - 20p$

91. $3m^3 - 21m^2 + 30m$

92. $11n^3 - 55n^2 + 44n$

94. $6y^4 + 12y^3 - 48y^2$

95. $2t^2 + 7t + 5$

97. $11x^2 + 34x + 3$

98. $7b^2 + 50b + 7$

100. $5x^2 - 17x + 6$

101. $4q^2 - 7q - 2$

103. $6p^2 - 19pq + 10q^2$

104. $21m^2 - 29mn + 10n^2$

106. $6u^2 + 5uv - 14v^2$

107. $-16x^2 - 32x - 16$

109. $-30q^3 - 140q^2 - 80q$

Factor Trinomials of the Form $ax^2 + bx + c$ using the ‘ac’ Method

In the following exercises, factor using the ‘ac’ method.

111. $5n^2 + 21n + 4$

113. $4k^2 - 16k + 15$

114. $5s^2 - 9s + 4$

116. $6p^2 + p - 22$

117. $2n^2 - 27n - 45$

119. $60y^2 + 290y - 50$

120. $6u^2 - 46u - 16$

122. $90n^3 + 42n^2 - 216n$

123. $16s^2 + 40s + 24$

125. $48y^2 + 12y - 36$

126. $30x^2 + 105x - 60$

Factor Using Substitution

In the following exercises, factor using substitution.

127. $x^4 - x^2 - 12$

129. $x^4 - 3x^2 - 28$

130. $x^4 - 13x^2 - 30$

132. $(x - 2)^2 - 3(x - 2) - 54$

Mixed Practice

In the following exercises, factor each expression using any method.

135. $u^2 - 12u + 36$

137. $r^2 - 20rs + 64s^2$

138. $q^2 - 29qr - 96r^2$

140. $12x^2 + 36y - 24z$

141. $6n^2 + 5n - 4$

143. $13z^2 + 39z - 26$

144. $5r^2 + 25r + 30$

146. $7x^2 - 21x$

147. $6r^2 + 30r + 36$

149. $24n^2 + 20n + 4$

150. $4a^2 + 5a + 2$

152. $x^4 - 7x^2 - 8$

153. $(x + 3)^2 - 9(x + 3) - 36$

6.3

Factor Special Products

Learning Objectives

By the end of this section, you will be able to:

- Factor perfect square trinomials
- Factor differences of squares
- Factor sums and differences of cubes

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $(3x^2)^3$.
2. Multiply: $(m+4)^2$.
3. Multiply: $(x-3)(x+3)$.

We have seen that some binomials and trinomials result from special products—squaring binomials and multiplying conjugates. If you learn to recognize these kinds of polynomials, you can use the special products patterns to factor them much more quickly.

Factor Perfect Square Trinomials

Some trinomials are perfect squares. They result from multiplying a binomial times itself. We squared a binomial using the Binomial Squares pattern in a previous chapter.

$$\left(\overset{a}{3x} + \overset{b}{4}\right)^2$$

$$\overset{a^2}{(3x)^2} + \overset{2 \cdot a \cdot b}{2(3x \cdot 4)} + \overset{b^2}{4^2}$$

$$9x^2 + 24x + 16$$

The trinomial $9x^2 + 24x + 16$ is called a *perfect square trinomial*. It is the square of the binomial $3x + 4$.

In this chapter, you will start with a perfect square trinomial and factor it into its prime factors.

You could factor this trinomial using the methods described in the last section, since it is of the form $ax^2 + bx + c$. But if you recognize that the first and last terms are squares and the trinomial fits the perfect square trinomials pattern, you will save yourself a lot of work.

Here is the pattern—the reverse of the binomial squares pattern.

Perfect Square Trinomials Pattern

If *a* and *b* are real numbers

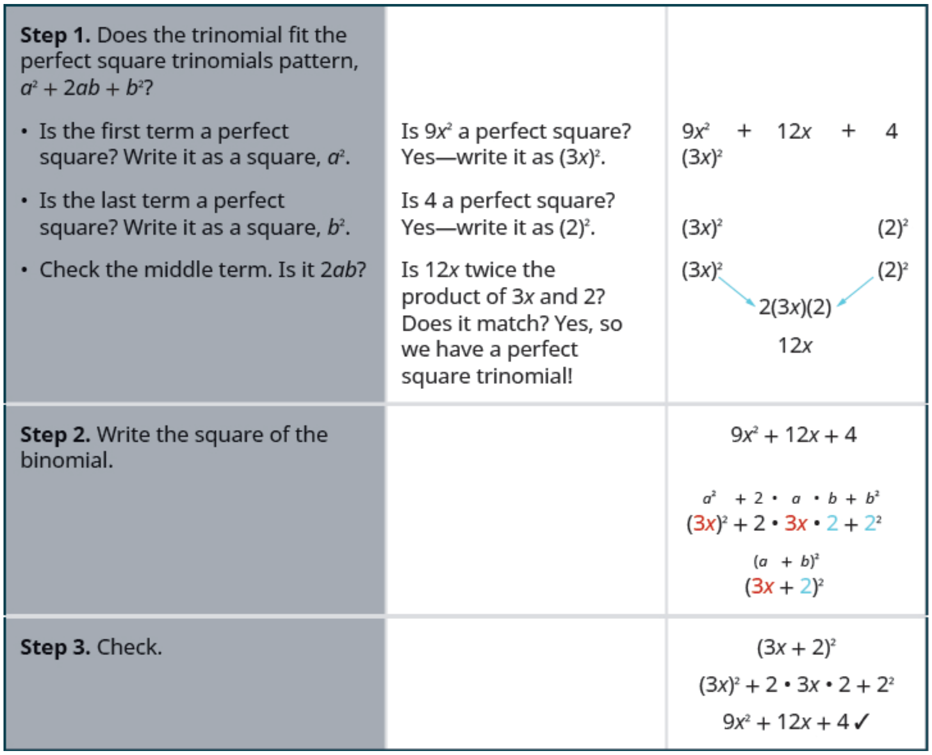
$$a^2 + 2ab + b^2 = (a+b)^2$$
$$a^2 - 2ab + b^2 = (a-b)^2$$

To make use of this pattern, you have to recognize that a given trinomial fits it. Check first to see if the leading coefficient is a perfect square, a^2. Next check that the last term is a perfect square, b^2. Then check the middle term—is it the product, $2ab$? If everything checks, you can easily write the factors.

EXAMPLE 6.23 HOW TO FACTOR PERFECT SQUARE TRINOMIALS

Factor: $9x^2 + 12x + 4$.

Solution



Step 1. Does the trinomial fit the perfect square trinomials pattern, $a^2 + 2ab + b^2$? • Is the first term a perfect square? Write it as a square, a^2. • Is the last term a perfect square? Write it as a square, b^2. • Check the middle term. Is it $2ab$?	Is $9x^2$ a perfect square? Yes—write it as $(3x)^2$. Is 4 a perfect square? Yes—write it as $(2)^2$. Is $12x$ twice the product of $3x$ and 2? Does it match? Yes, so we have a perfect square trinomial!	$9x^2 \quad + \quad 12x \quad + \quad 4$ $(3x)^2$ $(3x)^2 \qquad (2)^2$ $(3x)^2 \searrow 2(3x)(2) \swarrow (2)^2$ $12x$
Step 2. Write the square of the binomial.		$9x^2 + 12x + 4$ $a^2 + 2 \cdot a \cdot b + b^2$ $(3x)^2 + 2 \cdot 3x \cdot 2 + 2^2$ $(a + b)^2$ $(3x + 2)^2$
Step 3. Check.		$(3x + 2)^2$ $(3x)^2 + 2 \cdot 3x \cdot 2 + 2^2$ $9x^2 + 12x + 4$ ✓

TRY IT : : 6.45 Factor: $4x^2 + 12x + 9.$

TRY IT : : 6.46 Factor: $9y^2 + 24y + 16.$

The sign of the middle term determines which pattern we will use. When the middle term is negative, we use the pattern $a^2 - 2ab + b^2$, which factors to $(a - b)^2$.

The steps are summarized here.

HOW TO : : FACTOR PERFECT SQUARE TRINOMIALS.

Step 1.	Does the trinomial fit he pattern?	$a^2 + 2ab + b^2$	$a^2 - 2ab + b^2$
	Is the first term a perfect square?Write it as a square.	$(a)^2$	$(a)^2$
	Is the last term a perfect square? Write it as a square.	$(a)^2 \qquad (b)^2$	$(a)^2 \qquad (b)^2$
	Check the middle term. Is it $2ab$?	$(a)^2 \searrow 2 \cdot a \cdot b \swarrow (b)^2$	$(a)^2 \searrow 2 \cdot a \cdot b \swarrow (b)^2$
Step 2.	Write the square of the binomial.	$(a + b)^2$	$(a - b)^2$
Step 3.	Check by multiplying.		

We'll work one now where the middle term is negative.

EXAMPLE 6.24

Factor: $81y^2 - 72y + 16$.

Solution

The first and last terms are squares. See if the middle term fits the pattern of a perfect square trinomial. The middle term is negative, so the binomial square would be $(a - b)^2$.

	$81y^2 - 72y + 16$
Are the first and last terms perfect squares?	$(9y)^2 \qquad (4)^2$
Check the middle term.	$(9y)^2 \qquad (4)^2$ $2(9y)(4)$ $72y$
Does it match $(a - b)^2$? Yes.	$\overset{a^2}{(9y)^2} \overset{-\,2}{-\,2} \cdot \overset{a}{9y} \cdot \overset{b}{4} \overset{+}{+} \overset{b^2}{4^2}$
Write as the square of a binomial.	$(9y - 4)^2$
Check by multiplying:	

$$(9y - 4)^2$$
$$(9y)^2 - 2 \cdot 9y \cdot 4 + 4^2$$
$$81y^2 - 72y + 16 \checkmark$$

TRY IT : : 6.47 Factor: $64y^2 - 80y + 25$.

TRY IT : : 6.48 Factor: $16z^2 - 72z + 81$.

The next example will be a perfect square trinomial with two variables.

EXAMPLE 6.25

Factor: $36x^2 + 84xy + 49y^2$.

Solution

	$36x^2 + 84xy + 49y^2$
Test each term to verify the pattern.	$\overset{a^2}{(6x)^2} \overset{+\,2}{+\,2} \cdot \overset{a}{6x} \cdot \overset{b}{7y} \overset{+}{+} \overset{b^2}{(7y)^2}$
Factor.	$(6x + 7y)^2$
Check by multiplying.	

$$(6x + 7y)^2$$
$$(6x)^2 + 2 \cdot 6x \cdot 7y + (7y)^2$$
$$36x^2 + 84xy + 49y^2 \checkmark$$

TRY IT : : 6.49 Factor: $49x^2 + 84xy + 36y^2$.

TRY IT : : 6.50 Factor: $64m^2 + 112mn + 49n^2$.

Remember the first step in factoring is to look for a greatest common factor. Perfect square trinomials may have a GCF in all three terms and it should be factored out first. And, sometimes, once the GCF has been factored, you will recognize a perfect square trinomial.

EXAMPLE 6.26

Factor: $100x^2y - 80xy + 16y$.

Solution

	$100x^2y - 80xy + 16y$
Is there a GCF? Yes, $4y$, so factor it out.	$4y(25x^2 - 20x + 4)$
Is this a perfect square trinomial?	
Verify the pattern.	$4y\left[\overset{a^2}{(5x)^2} \overset{-\,2}{-\,2} \cdot \overset{a}{5x} \cdot \overset{b}{2} \overset{+}{+} \overset{b^2}{2^2}\right]$
Factor.	$4y(5x - 2)^2$

Remember: Keep the factor 4*y* in the final product.

Check:

$$\begin{gathered} 4y(5x-2)^2 \\ 4y\left[(5x)^2 - 2\cdot 5x\cdot 2 + 2^2\right] \\ 4y\left(25x^2 - 20x + 4\right) \\ 100x^2y - 80xy + 16y \checkmark \end{gathered}$$

TRY IT : : 6.51 Factor: $8x^2y - 24xy + 18y$.

TRY IT : : 6.52 Factor: $27p^2q + 90pq + 75q$.

Factor Differences of Squares

The other special product you saw in the previous chapter was the Product of Conjugates pattern. You used this to multiply two binomials that were conjugates. Here’s an example:

$$\overset{(a\ -\ b)}{(3x-4)}\overset{(a\ +\ b)}{(3x+4)}$$

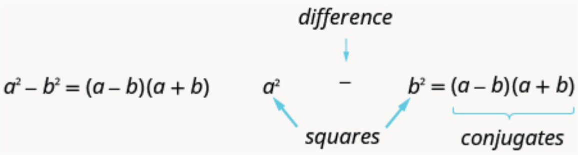
$$\overset{(a)^2}{(3x)^2} \overset{-}{-} \overset{(b)^2}{(4)^2}$$

$$9x^2 - 16$$

A difference of squares factors to a product of conjugates.

Difference of Squares Pattern

If *a* and *b* are real numbers,



Remember, “difference” refers to subtraction. So, to use this pattern you must make sure you have a binomial in which two squares are being subtracted.

EXAMPLE 6.27 HOW TO FACTOR A TRINOMIAL USING THE DIFFERENCE OF SQUARES

Factor: $64y^2 - 1$.

Solution

Step 1. Does the binomial fit the pattern?		$64y^2 - 1$
• Is this a difference?	Yes	$64y^2 - 1$
• Are the first and last terms perfect squares?	Yes	
Step 2. Write them as squares.	Write them as x^2 and 2^2.	$a^2 - b^2$ $(8y)^2 - 1^2$
Step 3. Write the product of conjugates.		$(a - b)(a + b)$ $(8y-1)(8y+1)$
Step 4. Check.		$(8y-1)(8y+1)$ $64y^2 - 1$ ✓

TRY IT : : 6.53 Factor: $121m^2 - 1$.

TRY IT : : 6.54 Factor: $81y^2 - 1$.

HOW TO : : FACTOR DIFFERENCES OF SQUARES.

Step 1.	Does the binomial fit the pattern?	$a^2 - b^2$
	Is this a difference?	$\_\_\_\_ - \_\_\_\_$
	Are the first and last terms perfect squares?	
Step 2.	Write them as squares.	$(a)^2 - (b)^2$
Step 3.	Write the product of conjugates.	$(a-b)(a+b)$
Step 4.	Check by multiplying.	

It is important to remember that *sums of squares do not factor into a product of binomials*. There are no binomial factors that multiply together to get a sum of squares. After removing any GCF, the expression $a^2 + b^2$ is prime!

The next example shows variables in both terms.

EXAMPLE 6.28

Factor: $144x^2 - 49y^2$.

Solution

	$144x^2 - 49y^2$
Is this a difference of squares? Yes.	$(12x)^2 - (7y)^2$
Factor as the product of conjugates.	$(12x - 7y)(12x + 7y)$
Check by multiplying.	

$$(12x - 7y)(12x + 7y)$$
$$144x^2 - 49y^2 \checkmark$$

TRY IT : : 6.55 Factor: $196m^2 - 25n^2$.

TRY IT : : 6.56 Factor: $121p^2 - 9q^2$.

As always, you should look for a common factor first whenever you have an expression to factor. Sometimes a common factor may “disguise” the difference of squares and you won’t recognize the perfect squares until you factor the GCF.

Also, to completely factor the binomial in the next example, we’ll factor a difference of squares twice!

EXAMPLE 6.29

Factor: $48x^4y^2 - 243y^2$.

Solution

	$48x^4y^2 - 243y^2$
Is there a GCF? Yes, $3y^2$—factor it out!	$3y^2(16x^4 - 81)$
Is the binomial a difference of squares? Yes.	$3y^2\left((4x^2)^2 - (9)^2\right)$
Factor as a product of conjugates.	$3y^2(4x^2 - 9)(4x^2 + 9)$
Notice the first binomial is also a difference of squares!	$3y^2((2x)^2 - (3)^2)(4x^2 + 9)$
Factor it as the product of conjugates.	$3y^2(2x - 3)(2x + 3)(4x^2 + 9)$

The last factor, the sum of squares, cannot be factored.

Check by multiplying:

$$3y^2(2x - 3)(2x + 3)(4x^2 + 9)$$
$$3y^2(4x^2 - 9)(4x^2 + 9)$$
$$3y^2(16x^4 - 81)$$
$$48x^4y^2 - 243y^2 \checkmark$$

TRY IT : : 6.57 Factor: $2x^4y^2 - 32y^2$.

TRY IT : : 6.58 Factor: $7a^4c^2 - 7b^4c^2$.

The next example has a polynomial with 4 terms. So far, when this occurred we grouped the terms in twos and factored from there. Here we will notice that the first three terms form a perfect square trinomial.

EXAMPLE 6.30

Factor: $x^2-6x+9-y^2$.

Solution

Notice that the first three terms form a perfect square trinomial.

	$x^2-6x+9-y^2$
Factor by grouping the first three terms.	$\underbrace{x^2-6x+9}-y^2$
Use the perfect square trinomial pattern.	$(x-3)^2-y^2$
Is this a difference of squares? Yes.	
Yes—write them as squares.	$\overset{a^2}{(x-3)^2}\overset{-b^2}{-y^2}$
Factor as the product of conjugates.	$(\overset{(a}{(x-3)}\overset{-b)}{-y})(\overset{(x}{(x-3)}\overset{+b)}{+y})$
	$(x-3-y)(x-3+y)$

You may want to rewrite the solution as $(x-y-3)(x+y-3)$.

TRY IT : : 6.59 Factor: $x^2-10x+25-y^2$.

TRY IT : : 6.60 Factor: $x^2+6x+9-4y^2$.

Factor Sums and Differences of Cubes

There is another special pattern for factoring, one that we did not use when we multiplied polynomials. This is the pattern for the sum and difference of cubes. We will write these formulas first and then check them by multiplication.

$$a^3+b^3=(a+b)\left(a^2-ab+b^2\right)$$
$$a^3-b^3=(a-b)\left(a^2+ab+b^2\right)$$

We'll check the first pattern and leave the second to you.

	$(a+b)(a^2-ab+b^2)$
Distribute.	$a(a^2-ab+b^2)+b(a^2-ab+b^2)$
Multiply.	$a^3-a^2b+ab^2+a^2b-ab^2+b^3$
Combine like terms.	a^3+b^3

Sum and Difference of Cubes Pattern

$$a^3+b^3=(a+b)\left(a^2-ab+b^2\right)$$
$$a^3-b^3=(a-b)\left(a^2+ab+b^2\right)$$

The two patterns look very similar, don't they? But notice the signs in the factors. The sign of the binomial factor matches the sign in the original binomial. And the sign of the middle term of the trinomial factor is the opposite of the sign in the original binomial. If you recognize the pattern of the signs, it may help you memorize the patterns.

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

same sign

opposite signs

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

same sign

opposite signs

The trinomial factor in the sum and difference of cubes pattern cannot be factored.

It be very helpful if you learn to recognize the cubes of the integers from 1 to 10, just like you have learned to recognize squares. We have listed the cubes of the integers from 1 to 10 in **Table 6.22**.

n	1	2	3	4	5	6	7	8	9	10
n^3	1	8	27	64	125	216	343	512	729	1000

Table 6.22

EXAMPLE 6.31 HOW TO FACTOR THE SUM OR DIFFERENCE OF CUBES

Factor: $x^3 + 64$.

Solution

Step 1. Does the binomial fit the sum or difference of cubes pattern? • Is it a sum or difference? • Are the first and last terms perfect cubes?	 This is a sum. Yes.	$x^3 + 64$ $x^3 + 64$
Step 2. Write the terms as cubes.	Write them as x^3 and 4^3.	$\underset{x^3 + 4^3}{a^3 + b^3}$
Step 3. Use either the sum or difference of cubes pattern.	This is a sum of cubes.	$\left(\underset{x+4}{a+b}\right)\left(\underset{x^2-4x+4^2}{a^2-ab+b^2}\right)$
Step 4. Simplify inside the parentheses.	It is already simplified.	$(x + 4)(x^2 - 4x + 16)$
Step 5. Check by multiplying the factors.		$\begin{array}{r} x^2 - 4x + 16 \\ x + 4 \\ \hline 4x^2 - 16x + 64 \\ x^3 - 4x^2 + 16x \\ \hline x^3 + 64 \end{array}$ ✓

TRY IT : : 6.61 Factor: $x^3 + 27$.

TRY IT : : 6.62 Factor: $y^3 + 8$.

HOW TO : : FACTOR THE SUM OR DIFFERENCE OF CUBES.

Step 1. Does the binomial fit the sum or difference of cubes pattern?
Is it a sum or difference?
Are the first and last terms perfect cubes?

Step 2. Write them as cubes.

Step 3. Use either the sum or difference of cubes pattern.

Step 4. Simplify inside the parentheses.

Step 5. Check by multiplying the factors.

EXAMPLE 6.32

Factor: $27u^3 - 125v^3$.

Solution

	$27u^3 - 125v^3$
This binomial is a difference. The first and last terms are perfect cubes.	
Write the terms as cubes.	$\overset{a^3}{(3u)^3} \overset{-}{-} \overset{b^3}{(5v)^3}$
Use the difference of cubes pattern.	$\left(\overset{a}{3u} \overset{-}{-} \overset{b}{5v}\right)\left(\overset{a^2}{(3u)^2} \overset{+}{+} \overset{ab}{3u \cdot 5v} \overset{+}{+} \overset{b^2}{(5v)^2}\right)$
Simplify.	$\left(\overset{a}{3u} \overset{-}{-} \overset{b}{5v}\right)\left(\overset{a^2}{9u^2} \overset{+}{+} \overset{ab}{15uv} \overset{+}{+} \overset{b^2}{25v^2}\right)$
Check by multiplying.	We'll leave the check to you.

TRY IT : : 6.63 Factor: $8x^3 - 27y^3$.

TRY IT : : 6.64 Factor: $1000m^3 - 125n^3$.

In the next example, we first factor out the GCF. Then we can recognize the sum of cubes.

EXAMPLE 6.33

Factor: $6x^3 y + 48y^4$.

Solution

	$6x^3y+48y^4$
Factor the common factor.	$6y(x^3+8y^3)$
This binomial is a sum The first and last terms are perfect cubes.	
Write the terms as cubes.	$6y\left(\overset{a^3}{x^3}\overset{+}{+}\overset{b^3}{(2y)^3}\right)$
Use the sum of cubes pattern.	$6y\left(\overset{a}{x}\overset{+}{+}\overset{b}{2y}\right)\left(\overset{a^2}{x^2}\overset{-}{-}\overset{ab}{x\cdot 2y}\overset{+}{+}\overset{b^2}{(2y)^2}\right)$
Simplify.	$6y(x+2y)(x^2-2xy+4y^2)$

Check:

To check, you may find it easier to multiply the sum of cubes factors first, then multiply that product by $6y$. We'll leave the multiplication for you.

TRY IT : : 6.65 Factor: $500p^3+4q^3$.

TRY IT : : 6.66 Factor: $432c^3+686d^3$.

The first term in the next example is a binomial cubed.

EXAMPLE 6.34

Factor: $(x+5)^3-64x^3$.

Solution

	$(x+5)^3-64x^3$
This binomial is a difference. The first and last terms are perfect cubes.	
Write the terms as cubes.	$\overset{a^3}{(x+5)^3}\overset{-}{-}\overset{b^3}{(4x)^3}$
Use the difference of cubes pattern.	$\left(\overset{a}{(x+5)}\overset{-}{-}\overset{b}{4x}\right)\left(\overset{a^2}{(x+5)^2}\overset{+}{+}\overset{a}{(x+5)}\cdot\overset{b}{4x}\overset{+}{+}\overset{b^2}{(4x)^2}\right)$
Simplify.	$(x+5-4x)(x^2+10x+25+4x^2+20x+16x^2)$
	$(-3x+5)(21x^2+30x+25)$
Check by multiplying.	We'll leave the check to you.

TRY IT : : 6.67 Factor: $(y+1)^3-27y^3$.

TRY IT : : 6.68 Factor: $(n+3)^3-125n^3$.

6.3 EXERCISES

Practice Makes Perfect

Factor Perfect Square Trinomials

In the following exercises, factor completely using the perfect square trinomials pattern.

159. $16y^2 + 24y + 9$

161. $36s^2 + 84s + 49$

162. $49s^2 + 154s + 121$

164. $64z^2 - 16z + 1$

165. $25n^2 - 120n + 144$

167. $49x^2 + 28xy + 4y^2$

168. $25r^2 + 60rs + 36s^2$

170. $64m^2 - 34m + 1$

171. $10jk^2 + 80jk + 160j$

173. $75u^4 - 30u^3v + 3u^2v^2$

Factor Differences of Squares

In the following exercises, factor completely using the difference of squares pattern, if possible.

175. $25v^2 - 1$

177. $4 - 49x^2$

178. $121 - 25s^2$

180. $98r^3 - 72r$

181. $24p^2 + 54$

183. $121x^2 - 144y^2$

184. $49x^2 - 81y^2$

186. $36p^2 - 49q^2$

187. $16z^4 - 1$

189. $162a^4b^2 - 32b^2$

190. $48m^4n^2 - 243n^2$

192. $p^2 + 14p + 49 - q^2$

Factor Sums and Differences of Cubes

In the following exercises, factor completely using the sums and differences of cubes pattern, if possible.

195. $x^3 + 125$

197. $z^6 - 27$

198. $v^3 - 216$

200. $125 - 27w^3$

201. $8y^3 - 125z^3$

203. $216a^3 + 125b^3$

204. $27y^3 + 8z^3$

206. $6x^3 - 48y^3$

207. $2x^2 - 16x^2 y^3$

209. $(x+3)^3 + 8x^3$

210. $(x+4)^3 - 27x^3$

212. $(y-5)^3 + 125y^3$

Mixed Practice

In the following exercises, factor completely.

213. $64a^2 - 25$

215. $27q^2 - 3$

216. $4p^2 - 100$

218. $36y^2 + 12y + 1$

219. $8p^2 + 2$

221. $125 - 8y^3$

222. $27u^3 + 1000$

224. $48q^3 - 24q^2 + 3q$

225. $x^2 - 10x + 25 - y^2$

227. $(x+1)^3 + 8x^3$